

LOWER BOUND FOR THE RATE OF BLOW-UP OF SINGULAR SOLUTIONS OF THE ZAKHAROV SYSTEM IN \mathbb{R}^3

J. COLLIANDER, M. CZUBAK, AND C. SULEM

ABSTRACT. We consider the scalar Zakharov system in \mathbb{R}^3 for initial conditions $(\psi(0), n(0), n_t(0)) \in H^{\ell+1/2} \times H^\ell \times H^{\ell-1}$, $0 \leq \ell \leq 1$. Assuming that the solution blows up in a finite time $t^* < \infty$, we establish a lower bound for the rate of blow-up of the corresponding Sobolev norms in the form

$$\|\psi(t)\|_{H^{\ell+1/2}} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} > C(t^* - t)^{-\theta_\ell}$$

with $\theta_\ell = \frac{1}{4}(1 + 2\ell)^-$. The analysis is a reappraisal of the local well-posedness theory of Ginibre, Tsutsumi and Velo (1997) combined with an argument developed by Cazenave and Weissler (1990) in the context of nonlinear Schrödinger equations.

1. INTRODUCTION

The Zakharov system describes the phenomenon of propagation of Langmuir waves in a non-magnetized plasma. It was derived by Zakharov [24] in the form of a coupled system governing the electric field complex amplitude $\psi(x, t)$ and the density fluctuations of ions $n(x, t)$. Here we consider the scalar Zakharov system in the form:

$$(1.1) \quad i\partial_t \psi + \Delta \psi = n\psi,$$

$$(1.2) \quad \partial_{tt} n - \Delta n = \Delta |\psi|^2,$$

where $\psi : (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{C}$, $n : (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$, with given initial conditions $\psi(0) = \psi_0$, $n(0) = n_0$ and $n_t(0) = n_1$.

There has been a large literature devoted to the local and global well-posedness of the initial value problem in the context of smooth solutions ([20], [1], [19], [17], [11]). In recent years, an effort has been made to lower the regularity assumptions ([6], [12], [18], [10], [3], [2], [22]) and to investigate the possible occurrence of local ill-posedness [13].

For general initial conditions in the energy space with negative Hamiltonian, solutions to the Zakharov system in two and three dimensions will

Date: May 3, 2013.

1991 Mathematics Subject Classification. 35Q55.

Key words and phrases. Zakharov system, local wellposedness, singular solutions, blow-up rate.

J.C. is partially supported by NSERC through grant number RGPIN 250233-12.

C.S. is partially supported by NSERC through grant number RGPIN 46179-11.

blow up in a finite or infinite time [16]. Heuristic arguments and numerical simulations show that solutions do blow up in a finite time both in two and three dimensions (see [21] for a review).

In two dimensions, there exist exact self-similar blow-up solutions [25]

$$(1.3) \quad \psi(x, t) = \frac{1}{a(t^* - t)} P\left(\frac{|x|}{a(t^* - t)}\right) e^{i\left(\theta + \frac{1}{a^2(t^* - t)} - \frac{|x|^2}{4(t^* - t)}\right)},$$

$$(1.4) \quad n(x, t) = \frac{1}{a^2(t^* - t)^2} N\left(\frac{|x|}{a(t^* - t)}\right),$$

where (P, N) satisfy the system of ODEs in the radial variable denoted η

$$(1.5) \quad \Delta P - P - NP = 0,$$

$$(1.6) \quad a^2(\eta^2 N_{\eta\eta} + 6\eta N_\eta + 6N) - \Delta N = \Delta P^2,$$

and $a > 0$ is a free parameter. Rigorous results on these solutions were proved in [11]. Numerical simulations show that for a large class of data, blow-up solutions asymptotically display a self-similar collapse described by the above solutions (1.3)-(1.4). In addition, Merle [15] established a lower bound for the rate of blow-up of singular solutions of the Zakharov system in the energy space in the form

$$(1.7) \quad \|u(t)\|_{H^1} \geq C(t^* - t)^{-1}, \quad \|n(t)\|_{L^2} \geq C(t^* - t)^{-1}.$$

This rate is optimal, in the sense that the exact solutions (1.3)-(1.4) solutions blow up exactly in this fashion. It is an open question whether there exist other solutions that blow up at a faster rate.

Merle uses a time-dependent rescaling based on the scale invariance of the wave equation. The scaling factor is associated to the energy norm and the energy conservation is interpreted in terms of the new variables. The optimal constant for Sobolev inequality expressed in terms of the ground state of the 2d cubic NLS equation is used to obtain a lower bound for the scaling factor, which in turn is related to the energy norm of the solution. A completely new element in [15] was a compactness argument leading to a limiting object as t approaches t^* . This method, now referred to as ‘the Liouville approach’ opened doors to break-through developments in the field.

The situation for the Zakharov system in three dimensions is more complex and several questions remain open. There are no known explicit blow-up solutions. Self-similar solutions exist only asymptotically close to collapse. They have the universal form [7], [25]

$$(1.8) \quad \psi(x, t) = \frac{1}{(t^* - t)} P\left(\frac{|x|}{(t^* - t)^{2/3}}\right) e^{i(t^* - t)^{-1/3}},$$

$$(1.9) \quad n(x, t) = \frac{1}{(t^* - t)^{4/3}} N\left(\frac{|x|}{(t^* - t)^{2/3}}\right),$$

where $P(\eta)$ and $N(\eta)$ are radially symmetric scalar functions satisfying the coupled system of ODEs

$$(1.10) \quad \Delta P - \frac{1}{3}P - NP = 0,$$

$$(1.11) \quad \frac{2}{9}(2\eta^2 N_{\eta\eta} + 13\eta N_\eta + 14N) = \Delta P^2.$$

Note that there is no free parameter in this system. In addition, unlike the 2d case, there is no rigorous proof of existence of solutions to the ODE system (1.10)-(1.11). The profile system (1.10)-(1.11) was studied numerically in [25] where two pairs of localized solutions were computed; one of them displaying a monotone profile for P and N . Like for the 2d problem, numerical simulations of the three-dimensional Zakharov system [14] show that, for a large class of initial conditions, solutions blow up in a finite time and display a self-similar collapse described by (1.8)-(1.9). This can be seen as the dynamic stability of these asymptotic solutions. Asymptotically close to the collapse, the regime is strongly supersonic with the pressure term Δn negligible compared to the ion-inertia term in (1.2).

In the present note, we consider the question of the rate of blow-up of solutions for the Zakharov system in three dimensions, and we establish a lower bound for it in appropriate Sobolev norms. Our method differs from that developed by Merle [15] for the 2d problem. It is in a sense simpler but less precise. The two main ingredients are a local well-posedness result and a contradiction argument adapted from Cazenave and Weissler [9].

The notion of criticality plays a central role in the study of the Nonlinear Schrödinger equation (NLS). For the Zakharov system however, criticality is less straightforward because the NLS and the wave equation scale differently. In [12], Ginibre, Tsutsumi and Velo proposed a definition of criticality for the Zakharov system for initial condition $(\psi(0), n(0), n_t(0))$ in $H^k \times H^\ell \times H^{\ell-1}$ with the critical values being $k = d/2 - 3/2$, $\ell = d/2 - 2$, and d is the spatial dimension. Note that $k - \ell = 1/2$ and not one as the energy space would suggest.

In three dimensions, the critical space in the above sense is $L^2 \times H^{-1/2} \times H^{-1}$ which is, up to $\epsilon > 0$ the space in which Bejenaru and Herr [2] recently proved local well-posedness. Also, the asymptotic solution (1.8)-(1.9) has the property that the H^k norm of ψ and the H^ℓ norm of n blow up at the same rate when $k - \ell = 1/2$.

Our analysis relies on the local well-posedness results of Ginibre et al [12] in $H_\ell = H^{\ell+1/2} \times H^\ell \times H^{\ell-1}$, $\ell \geq 0$, thus concerns solutions that are slightly more regular than solutions in the critical space.

Theorem 1.1. *Consider initial conditions $(\psi(0), n(0), n_t(0))$ in H_ℓ , $0 \leq \ell \leq 1$. Assume that the solution (u, n) blows up in a finite time t^* in H_1 . Then, we have the lower bound for the rate of blow-up in the corresponding Sobolev norms*

$$(1.12) \quad \|\psi(t)\|_{H^{\ell+1/2}} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} > C(t^* - t)^{-\theta_\ell}$$

with $\theta_\ell = \frac{1}{4}(1 + 2\ell)^-$.

Remark 1.2. *We do not know whether this bound is optimal. In particular, we observe that the homogeneous $\dot{H}^{\ell+1/2}$ norm of ψ and the homogeneous \dot{H}^ℓ norm of n in the expression of the asymptotic solution (1.8)-(1.9) both blow up at a faster rate, namely $\frac{1}{3}(1 + 2\ell)$. However, as (1.8)-(1.9) is not a solution of (1.1)-(1.2), but only an asymptotic solution, its rate of blow-up might not be a real threshold. Moreover, for the cubic NLS in 3D, the present method gives a rate of blow-up of the H^1 norm to be $\frac{1}{4}$ [9], which has been observed in numerical simulations [21, Chapter 7].*

Remark 1.3. *We choose to consider the $H^{\ell+1/2} \times H^\ell \times H^{\ell-1}$ functional framework instead of the more general norm $H^k \times H^\ell \times H^{\ell-1}$, because for $k = l + \frac{1}{2}$, $\|\psi(t)\|_{\dot{H}^k}$ scales the same as $\|n(t)\|_{\dot{H}^\ell}$ when ψ and n are given by (1.8)-(1.9).*

Here is a brief description of the content of the paper. In Section 2, we recall important linear estimates. Section 3 is devoted to nonlinear estimates. In particular, we carefully keep track of the power of time involved in the estimates as it is central for the analysis of the lower bound for the blow-up rate. In Section 4, we adapt an argument for semilinear heat equations due to Weissler [23] and later extended to nonlinear Schrödinger equations by Cazenave and Weissler [9] to obtain a lower bound of blow-up for Sobolev norms of the solution.

2. PRELIMINARY ESTIMATES

Consider the Zakharov system (1.1)-(1.2) with initial conditions

$$(2.1) \quad (\psi, n, n_t)|_{t=0} = (\psi_0, n_0, n_1).$$

The wave equation (1.2) can be transformed into a reduced wave equation [3], [12] for

$$w = n + i\langle \nabla \rangle^{-1} \partial_t n,$$

where $\langle \nabla \rangle = (1 - \Delta)^{1/2}$.

The new system then takes the form

$$(2.2) \quad i\partial_t \psi + \Delta \psi = (\mathcal{R}e \ w)\psi,$$

$$(2.3) \quad (i\partial_t - \langle \nabla \rangle)w = -\langle \nabla \rangle^{-1} \Delta |\psi|^2 - \langle \nabla \rangle^{-1} \mathcal{R}e \ w,$$

and (ψ, w) solve (2.2)-(2.3) with data $(\psi_0, w_0) = (\psi_0, n_0 + i\langle \nabla \rangle^{-1} n_1)$ if and only if $(\psi, \mathcal{R}e \ w)$ solve (1.1)-(1.2) with data (ψ_0, n_0, n_1) .

We will use space-time norms in the context of solutions defined on a finite time interval $(-T, T)$, and we introduce an even time cut-off function $\varphi \in C^\infty$ satisfying $\varphi(t) = 1$ for $|t| \leq 1$, $\varphi(t) = 0$ for $|t| \geq 2$, $0 \leq \varphi(t) \leq 1$. We denote $\varphi_T(t) = \varphi(t/T)$, $(T \leq 1)$. The Duhamel representation of the

solution takes the form

$$(2.4) \quad \psi(t) = \varphi_1(t)U(t)\psi_0 - i\varphi_T(t) \int_0^t U(t-s)f_1(s)ds,$$

$$(2.5) \quad w(t) = \varphi_1(t)W(t)w_0 + i\varphi_T(t) \int_0^t W(t-s) \left(f(s) + \varphi_{2T} \frac{\mathcal{R}e w}{\langle \nabla \rangle} \right) ds,$$

where

$$U(t) = e^{it\Delta}, \quad W(t) = e^{-it\sqrt{1-\Delta}}, \\ f_1 = \varphi_{2T}^2 (\mathcal{R}e w)\psi, \quad f = \langle \nabla \rangle^{-1} \varphi_{2T}^2 \Delta |\psi|^2.$$

Building on the foundation established in [4] and following [12], we seek a solution $(\psi, w) \in X_S^{l+\frac{1}{2},b} \times X_W^{l,b}$, which are the space-time weighted Bourgain spaces with norms respectively given by

$$\|\psi\|_{X_S^{l+\frac{1}{2},b}} = \left\| \langle \xi \rangle^{l+\frac{1}{2}} \langle \tau + |\xi|^2 \rangle^b \hat{\psi}(\tau, \xi) \right\|_{L_{\tau,\xi}^2}, \\ \|w\|_{X_W^{l,b}} = \left\| \langle \xi \rangle^l \langle \tau + |\xi| \rangle^b \hat{w}(\tau, \xi) \right\|_{L_{\tau,\xi}^2},$$

where we use the notation $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Note the difference in the dispersive weights for the above two norms. We are using $\langle \tau + |\xi| \rangle$ for the reduced wave equation, which is equivalent to $\langle \tau + \langle \xi \rangle \rangle$. Also, we did not find a benefit of using two different b indices.

We now recall important linear estimates from [12] (see also [3]).

Lemma 2.1. *Let $s, b \in \mathbb{R}$ and $(X^{s,b}, \tilde{\mathbf{U}}(\mathbf{t})) = (X_S^{s,b}, U(t))$ or $(X^{s,b}, \tilde{\mathbf{U}}(\mathbf{t})) = (X_W^{s,b}, W(t))$. Then*

$$(2.6) \quad \|\varphi_1 \tilde{\mathbf{U}}(t)u_0\|_{X^{s,b}} = \|\varphi_1\|_{H^b} \|u_0\|_{H^s}.$$

Let $-1/2 < b' \leq 0 \leq b \leq b' + 1$, and $T \leq 1$. Then

$$(2.7) \quad \|\varphi_T \int_0^t \tilde{\mathbf{U}}(t-t')f(s)dt'\|_{X^{s,b}} \leq CT^{1-b+b'} \|f\|_{X^{s,b'}}.$$

The cut-off function φ_{2T} has been introduced inside the nonlinear term in (2.4)-(2.5). Its effect is evaluated in the next lemma.

Lemma 2.2. *For any $s \in \mathbb{R}$, $b \geq 0$, $q \geq 2$ and $bq > 1$,*

$$(2.8) \quad \|\varphi_T f\|_{X^{s,b}} \leq CT^{-b+1/q} \|f\|_{X^{s,b}},$$

where $X^{s,b} = X_S^{s,b}$ or $X^{s,b} = X_W^{s,b}$.

Note that for a parameter $b > 1/2$, the negative power of T is minimized with $q = 2$.

We apply Lemma 2.1 to (2.4) with

$$b' = b - 1 + \epsilon, \quad 0 < \epsilon \ll 1,$$

and obtain

$$(2.9) \quad \|\psi\|_{X_S^{\ell+\frac{1}{2},b}} \lesssim \|\psi_0\|_{H^{\ell+\frac{1}{2}}} + T^\epsilon \|\varphi_{2T}^2(\mathcal{R}e w)\psi\|_{X_S^{\ell+\frac{1}{2},b-1+\epsilon}}.$$

Similarly for (2.5), we first apply Lemma 2.1 to the nonlinear term f with $b' = b - 1 + \epsilon$, $0 < \epsilon \ll 1$ and then to the linear term with $b' = 0$. This results in

$$(2.10) \quad \begin{aligned} & \|w\|_{X_W^{\ell,b}} \lesssim \|w_0\|_{H^\ell} \\ & + T^\epsilon \left\| \varphi_{2T}^2 \frac{\Delta}{\langle \nabla \rangle} |\psi|^2 \right\|_{X_W^{\ell,b-1+\epsilon}} + T^{1-b} \|\varphi_{2T} \langle \nabla \rangle^{-1} \mathcal{R}e w\|_{X_W^{\ell,0}}. \end{aligned}$$

The next section is dedicated to showing we can handle the nonlinearities on the right hand side, and produce *additional* powers of T in the process.

3. NONLINEAR ESTIMATES

We need to estimate the right hand side of (2.9) and (2.10), namely

$$f_1 = \varphi_{2T}^2(\mathcal{R}e w)\psi \in X_S^{k,-c}, \quad f = \langle \nabla \rangle^{-1} \varphi_{2T}^2 \Delta |\psi|^2 \in X_W^{\ell,-c},$$

where $c = -(b - 1 + \epsilon)$, and $k - \ell = \frac{1}{2}$. The second term in (2.10) will be treated in the next section. More precisely, we need to establish

$$(3.1) \quad \|\varphi_{2T}^2(\mathcal{R}e w)\psi\|_{X_S^{k,-c}} \lesssim T^\theta \|\varphi_{2T} \mathcal{R}e w\|_{X_W^{\ell,b}} \|\varphi_{2T} \psi\|_{X_S^{k,b}},$$

$$(3.2) \quad \|\langle \nabla \rangle^{-1} (\varphi_{2T}^2 \Delta |\psi|^2)\|_{X_W^{\ell,-c}} \lesssim T^\theta \|\varphi_{2T} \psi\|_{X_S^{k,b}}^2.$$

The following setup is standard. Let $\varphi_{2T} \mathcal{R}e w = u$. We consider the nonlinearities on the Fourier side following the notation of [12].

$$(3.3) \quad \hat{f}_1(\xi_1, \tau_1) = \int_{\mathbb{R}^{3+1}} \hat{u}(\xi_1 - \xi_2, \tau_1 - \tau_2) \widehat{(\varphi_{2T}^2 \psi)}(\xi_2, \tau_2) d\xi_2 d\tau_2,$$

$$(3.4) \quad |\hat{f}(\xi, \tau)| \leq |\xi| \int_{\mathbb{R}^{3+1}} |\widehat{(\varphi_{2T}^2 \psi)}(\xi + \xi_2, \tau + \tau_2)| |\widehat{(\varphi_{2T}^2 \psi)}(-\xi_2, -\tau_2)| d\xi_2 d\tau_2,$$

where we changed variables in the second integral, and used the trivial estimate $|\xi|^2 \langle \xi \rangle^{-1} \leq |\xi|$.

To estimate f_1 in $X_S^{k,-c}$, we define

$$(3.5) \quad \hat{v}_2(\xi_2, \tau_2) = \langle \xi_2 \rangle^k \langle \tau_2 + |\xi_2|^2 \rangle^b \widehat{(\varphi_{2T}^2 \psi)}(\xi_2, \tau_2),$$

$$(3.6) \quad \hat{v}(\xi, \tau) = \langle \xi \rangle^l \langle \tau + |\xi|^2 \rangle^b \hat{u}(\xi, \tau),$$

and by duality, take the scalar product with a test function in $X_S^{-k,c}$ or equivalently with a function whose Fourier transform is $\langle \xi_1 \rangle^k \langle \tau_1 + |\xi_1|^2 \rangle^{-c} \hat{v}_1(\xi_1, \tau_1)$ with v_1 in L_{xt}^2 .

Proceeding similarly for f , we are led to showing two estimates

$$(3.7) \quad |N_1(v, v_1, v_2)| \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2,$$

$$(3.8) \quad |N_2(v, v_1, v_2)| \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2,$$

where

$$(3.9) \quad N_1(v, v_1, v_2) = \int \frac{\hat{v} \hat{v}_1 \hat{v}_2 \langle \xi_1 \rangle^k d\xi_1 d\xi_2 d\tau_1 d\tau_2}{\langle \tau + |\xi| \rangle^b \langle \tau_1 + |\xi_1|^2 \rangle^c \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_2 \rangle^k \langle \xi \rangle^l},$$

$$(3.10) \quad N_2(v, v_1, v_2) = \int \frac{\hat{v} \hat{v}_1 \hat{v}_2 |\xi| \langle \xi \rangle^\ell d\xi_1 d\xi_2 d\tau_1 d\tau_2}{\langle \tau + |\xi| \rangle^c \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_1 \rangle^k \langle \xi_2 \rangle^k}.$$

The arguments of \hat{v} , \hat{v}_1 , \hat{v}_2 are (ξ, τ) , (ξ_1, τ_1) , (ξ_2, τ_2) with $\xi = \xi_1 - \xi_2$, $\tau = \tau_1 - \tau_2$, and we can also assume $\mathcal{F}^{-1}(\frac{\hat{v}}{\langle \tau + |\xi| \rangle^{\gamma a}})$, $\mathcal{F}^{-1}(\frac{\hat{v}_i}{\langle \tau_i + |\xi_i|^2 \rangle^{\gamma a_i}})$, $i = 1, 2$, have support in $|t| \leq CT$.

Introducing the variables $\sigma_i = \tau_i + |\xi_i|^2$ and $\sigma = \tau + |\xi|$, we can express

$$\xi_1^2 - \xi_2^2 - |\xi| = \sigma_1 - \sigma_2 - \sigma,$$

from which one concludes (using ideas first observed in [5] and used in [6] and [12, Lemma 3.3])

$$(3.11) \quad \langle \xi_1 \rangle^2 \lesssim \langle \sigma_1 \rangle + \langle \sigma_2 \rangle + \langle \sigma \rangle \quad \text{when} \quad |\xi_1| \geq 2|\xi_2|.$$

In [12], the proof of the two estimates in the range of exponents we are interested in, was accomplished in two lemmas (Lemma 3.4 and 3.5), which were obtained from a repeated application of a general estimate (shown in Lemma 3.2). The analysis of [12] did not require an optimal power of θ , but needed it to be just large enough, so the final power of T , after combining all the estimates, was positive. Hence it appears that certain simplifications were made, which resulted in cleaner estimates (See for example [12, Remark 3.1]. Here we seek the optimal power of θ , so we reprove estimates (3.7)-(3.8) with a goal to optimize the final power of θ . First we state a lemma that follows directly from [12, Lemma 3.2].

Lemma 3.1. *Let b_0, γ, a, a_1, a_2 satisfy*

$$(3.12) \quad b_0 > \frac{1}{2},$$

$$(3.13) \quad 0 \leq \gamma \leq 1,$$

$$(3.14) \quad a, a_1, a_2 \geq 0,$$

$$(3.15) \quad (1 - \gamma) \max(a, a_1, a_2) \leq b_0 \leq (1 - \gamma)(a + a_1 + a_2),$$

$$(3.16) \quad (1 - \gamma)a < b_0,$$

$$(3.17) \quad m \geq \frac{5}{2} - (1 - \gamma)(a + a_1 + a_2)/b_0 \geq 0,$$

with strict inequality¹ in (3.17L) if equality holds in (3.15R) or if $a_1 = 0$. And if

$$(3.18) \quad \frac{1}{2} > \gamma a, \gamma a_1, \gamma a_2,$$

¹We will not need to consider this case.

and $v, v_1, v_2 \in L^2(\mathbb{R}^3)$ are such that $\mathcal{F}^{-1}(\frac{\hat{v}}{\langle \tau + |\xi| \rangle^{\gamma a}}), \mathcal{F}^{-1}(\frac{\hat{v}_i}{\langle \tau_i + |\xi_i|^2 \rangle^{\gamma a_i}}), i = 1, 2$, have support in $|t| \leq CT$, then

$$(3.19) \quad \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^a \langle \tau_1 + |\xi_1|^2 \rangle^{a_1} \langle \tau_2 + |\xi_2|^2 \rangle^{a_2} \langle \xi \rangle^m} \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2,$$

$$(3.20) \quad \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^a \langle \tau_1 + |\xi_1|^2 \rangle^{a_1} \langle \tau_2 + |\xi_2|^2 \rangle^{a_2} \langle \xi_2 \rangle^m} \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2,$$

where

$$(3.21) \quad \theta = \gamma(a + a_1 + a_2).$$

Remark 3.2. Lemma 3.1 is a three dimensional version of [12, Lemma 3.2], and the only difference is that $\frac{1}{2}$ upper bound in (3.18) does not appear in [12]. We included it here to maximize the value of θ appearing in (3.21). It also results in a simpler formula in (3.21). For the general formula for θ see [12] [(3.24) and (3.14)].

We start by estimating N_1 and turn our attention to N_2 later.

Lemma 3.3. Let $0 \leq \ell \leq 1$ and let $\epsilon_0, \epsilon, \bar{\epsilon} > 0$ be sufficiently small. Suppose the functions v, v_1, v_2 satisfy the conditions of Lemma 3.1 and $b = \frac{1}{2} + \bar{\epsilon}$ and $c = 1 - \epsilon - b$, then the estimate (3.7) holds with

$$(3.22) \quad \theta = b + 1 - \epsilon - \left(\frac{5}{2} - \ell\right)\left(\frac{1}{2} + \epsilon_0\right).$$

The proof is an application of Lemma 3.1 and follows [12, Lemma 3.4], but again, we attempt to maximize θ .

Proof. Consider two regions

$$\text{Region 1: } \{|\xi_1| \leq 2|\xi_2|\},$$

$$\text{Region 2: } \{|\xi_1| > 2|\xi_2|\}.$$

In Region 1, (3.7) reduces to

$$\int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^b \langle \tau_1 + |\xi_1|^2 \rangle^c \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi \rangle^\ell} \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2,$$

which is exactly (3.19) with

$$(a, a_1, a_2, m) = (b, c, b, \ell).$$

Therefore the estimate follows if we can find $0 \leq \gamma \leq 1$ and b_0 such that the conditions (3.12)-(3.18) are satisfied. Let $b_0 = \frac{1}{2} + \epsilon_0$. One can check that if $\gamma = 1 - \left(\frac{5}{2} - \ell\right)\frac{b_0}{2b+c}$, then the conditions are satisfied, and we obtain

$$\theta = \gamma(2b + c) = b + 1 - \epsilon - \left(\frac{5}{2} - \ell\right)b_0,$$

as needed.

Now we consider Region 2. Here we use (3.11) to bound N_1 as follows

$$N_1 \lesssim I + I_1 + I_2,$$

where

$$\begin{aligned} I &= \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^{b-\frac{k-\ell}{2}} \langle \tau_1 + |\xi_1|^2 \rangle^c \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_2 \rangle^k}, \\ I_1 &= \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^b \langle \tau_1 + |\xi_1|^2 \rangle^{c-\frac{k-\ell}{2}} \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_2 \rangle^k}, \\ I_2 &= \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^b \langle \tau_1 + |\xi_1|^2 \rangle^c \langle \tau_2 + |\xi_2|^2 \rangle^{b-\frac{k-\ell}{2}} \langle \xi_2 \rangle^k}. \end{aligned}$$

We apply estimate (3.20) of Lemma 3.1 three times. Each time, we use the same value of γ' , $0 \leq \gamma' \leq 1$, which is chosen so that the resulting θ is the same in both regions. This means

$$\gamma' = \gamma \frac{2b+c}{2b+c-\frac{k-\ell}{2}} = \gamma \frac{2b+c}{2b+c-\frac{1}{4}}.$$

We let

$$\begin{aligned} (a, a_1, a_2, m) &= (b - \frac{1}{4}, c, b, k) \\ (a, a_1, a_2, m) &= (b, c - \frac{1}{4}, b, k) \\ (a, a_1, a_2, m) &= (b, c, b - \frac{1}{4}, k), \end{aligned}$$

and again one readily checks the conditions hold. In particular, in each of the above three cases, when we verify (3.15)R, it reduces to requiring

$$b_0 \leq (1-\gamma)(c+2b) - \frac{1}{4} = (\frac{5}{2} - \ell)b_0 - \frac{1}{4},$$

which holds if and only if $\ell \leq \frac{3}{2} - \frac{1}{4b_0}$. \square

We turn our attention to treating N_2 .

Lemma 3.4. *Let $0 \leq \ell \leq 1$, and let $\epsilon_0, \bar{\epsilon}, \epsilon_1 > 0$ be sufficiently small. Suppose the functions v, v_1, v_2 satisfy the conditions of Lemma 3.1 and $b = \frac{1}{2} + \bar{\epsilon}$ and $c = 1 - \epsilon - b$, then the estimate (3.8) holds with*

$$(3.23) \quad \theta = b + 1 - \epsilon - (\frac{5}{2} - \ell)b_0.$$

Note, θ here is the same as θ in Lemma 3.3. The general idea of the proof is the same as in Lemma 3.3 (also compare with [12, Lemma 3.5]). We include the details for completeness.

Proof. The proof is done in three regions, but due to symmetry of the estimate it is enough to consider only Region 1 and Region 2:

$$\begin{aligned} \text{Region 1: } & \left\{ \frac{|\xi_2|}{2} \leq |\xi_1| \leq 2|\xi_2| \right\}, \\ \text{Region 2: } & \{|\xi_1| > 2|\xi_2|\}. \\ \text{Region 3: } & \left\{ |\xi_1| \leq \frac{1}{2}|\xi_2| \right\}. \end{aligned}$$

In Region 1, (3.8) reduces to

$$\int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^c \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi \rangle^{2k-(\ell+1)}} \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2,$$

which is exactly (3.19) with

$$(a, a_1, a_2, m) = (c, b, b, \ell),$$

since $k = l + \frac{1}{2}$. Here, if we also let $\gamma = 1 - (\frac{5}{2} - \ell) \frac{b_0}{c+2b}$, then the estimate follows with θ given by (3.23).

In Region 2 we again use (3.11) to obtain

$$N_2 \lesssim I + I_1 + I_2,$$

where

$$\begin{aligned} I &= \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^{c-\frac{1}{4}} \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_2 \rangle^k}, \\ I_1 &= \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^c \langle \tau_1 + |\xi_1|^2 \rangle^{b-\frac{1}{4}} \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_2 \rangle^k}, \\ I_2 &= \int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \tau + |\xi| \rangle^c \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^{b-\frac{1}{4}} \langle \xi_2 \rangle^k}. \end{aligned}$$

As before, we apply Lemma 3.1 and use estimate (3.20) three times with the same $0 \leq \gamma' \leq 1$, which we choose so the resulting θ is the same in both regions. Hence

$$\gamma' = \gamma \frac{c+2b}{c+2b-\frac{1}{4}},$$

and

$$\begin{aligned} (a, a_1, a_2, m) &= (c - \frac{1}{4}, b, b, k) \\ (a, a_1, a_2, m) &= (c, b - \frac{1}{4}, b, k) \\ (a, a_1, a_2, m) &= (c, b, b - \frac{1}{4}, k). \end{aligned}$$

□

4. LOWER BOUND FOR THE RATE OF BLOW-UP OF SINGULAR SOLUTIONS

We first summarize a priori estimates derived in previous sections. Since (3.7)-(3.8) imply (3.1)-(3.2) combining with (2.9)-(2.10) we have

$$\begin{aligned}\|\psi\|_{X_S^{k,b}} &\lesssim \|\psi_0\|_{H^k} + T^{\epsilon+\theta} \|\varphi_{2T} \mathcal{R}e w\|_{X_W^{\ell,b}} \|\varphi_{2T} \psi\|_{X_S^{k,b}}, \\ \|w\|_{X_W^{\ell,b}} &\lesssim \|w_0\|_{H^\ell} + T^{\epsilon+\theta} \|\varphi_{2T} \psi\|_{X_S^{k,b}}^2 + T^{1-b} \|\varphi_{2T} \mathcal{R}e w\|_{X_W^{\ell,0}}.\end{aligned}$$

Next applying (2.8) with $q = 2$ we have in particular

$$T^{1-b} \|\varphi_{2T} \mathcal{R}e w\|_{X_W^{\ell,0}} \leq T^{1-b} \|\varphi_{2T} \mathcal{R}e w\|_{X_W^{\ell,b}} \leq T^{3/2-2b} \|\mathcal{R}e w\|_{X_W^{\ell,b}},$$

and hence

$$(4.1) \quad \|\psi\|_{X_S^{k,b}} \lesssim \|\psi_0\|_{H^k} + T^{\theta_\ell} \|\mathcal{R}e w\|_{X_W^{\ell,b}} \|\psi\|_{X_S^{k,b}}$$

$$(4.2) \quad \|w\|_{X_W^{\ell,b}} \lesssim \|w_0\|_{H^\ell} + T^{\theta_\ell} \|\psi\|_{X_S^{k,b}}^2 + T^{3/2-2b} \|w\|_{X_W^{\ell,b}},$$

where

$$\theta_\ell = \epsilon + c + 2b - \left(\frac{5}{2} - \ell\right)\left(\frac{1}{2} + \epsilon_0\right) - 2b + 1.$$

With the choice of $c = 1 - b - \epsilon = \frac{1}{2} - \bar{\epsilon} - \epsilon$ this reduces to

$$(4.3) \quad \theta_\ell = \frac{1}{4} + \frac{\ell}{2} - \frac{5}{2}\epsilon_0 + \ell\epsilon_0 - \bar{\epsilon} = \left(\frac{1}{4} + \frac{\ell}{2}\right)^-.$$

The two inequalities are rewritten as

$$\begin{aligned}\|\psi\|_{X_S^{k,b}} + \|w\|_{X_W^{\ell,b}} &\lesssim \|\psi_0\|_{H^k} + \|w_0\|_{H^\ell} \\ &\quad + T^{\theta_\ell} (\|\psi\|_{X_S^{k,b}} + \|w\|_{X_W^{\ell,b}})^2 + T^{3/2-2b} \|w\|_{X_W^{\ell,b}},\end{aligned}$$

or by using

$$\|w\|_{X_W^{\ell,b}} \sim \|n\|_{X_W^{\ell,b}} + \|n_t\|_{X_W^{\ell-1,b}} \quad \text{and} \quad \|w_0\|_{H^\ell} \sim \|n_0\|_{H^\ell} + \|n_1\|_{H^{\ell-1}}$$

equivalently we have

$$\begin{aligned}\|\psi\|_{X_S^{k,b}} + \|n\|_{X_W^{\ell,b}} + \|n_t\|_{X_W^{\ell-1,b}} &\leq C(\|\psi_0\|_{H^k} + \|n_0\|_{H^\ell} + \|n_1\|_{H^{\ell-1}}) \\ &\quad + CT^{\theta_\ell} (\|\psi\|_{X_S^{k,b}} + \|n\|_{X_W^{\ell,b}} + \|n_t\|_{X_W^{\ell-1,b}})^2 \\ (4.4) \quad &\quad + CT^{3/2-2b} (\|n\|_{X_W^{\ell,b}} + \|n_t\|_{X_W^{\ell-1,b}}).\end{aligned}$$

Further, we can assume T is small enough so that $CT^{3/2-2b} < 1/2$. Rearranging (4.4) we obtain

$$\begin{aligned}\|\psi\|_{X_S^{k,b}} + \|n\|_{X_W^{\ell,b}} + \|n_t\|_{X_W^{\ell-1,b}} &\leq 2C(\|\psi_0\|_{H^k} + \|n_0\|_{H^\ell} + \|n_1\|_{H^{\ell-1}}) \\ (4.5) \quad &\quad + 2CT^{\theta_\ell} (\|\psi\|_{X_S^{k,b}} + \|n\|_{X_W^{\ell,b}} + \|n_t\|_{X_W^{\ell-1,b}})^2.\end{aligned}$$

Then denoting

$$\mathcal{X}(M, T) = \{(\psi, n) : (\psi, n, n_t)|_{t=0} = (\psi_0, n_0, n_1), \\ \|\psi\|_{X_S^{k,b_1}} + \|n\|_{X_W^{\ell,b}} + \|n_t\|_{X_W^{\ell-1,b}} \leq M\},$$

the local wellposedness theory is obtained by a contraction argument in \mathcal{X} if $2CT^{\theta_\ell}M < 1$ and $CT^{3/2-2b} < 1/2$.

We adapt arguments developed Cazenave and Weissler [23], [9] to prove a lower bound on the rate of blow-up. Denote by t^* the supremum of all $T > 0$ for which there exists a solution (ψ, n) of the Zakharov system satisfying

$$\|\chi_{[0,T]}\psi\|_{X_S^{k,b}} + \|\chi_{[0,T]}n\|_{X_W^{\ell,b}} + \|\chi_{[0,T]}n_t\|_{X_W^{\ell-1,b}} < \infty.$$

The local well-posedness theory shows $t^* > 0$ and for all $t \in [0, t^*)$

$$\|\psi(t)\|_{H^k} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} < \infty.$$

By maximality of t^* , it is impossible that

$$\|\psi(t)\|_{L_{[0,t^*]}^\infty H^k} + \|n(t)\|_{L_{[0,t^*]}^\infty H^\ell} + \|n_t(t)\|_{L_{[0,t^*]}^\infty H^{\ell-1}} < \infty.$$

Otherwise, the initial value problem at time t^* with Cauchy data $\psi(t^*), n(t^*), n_t(t^*)$ would be well-defined and the local theory would provide an extension of the solution beyond t^* . Therefore, if $t^* < \infty$, blow-up occurs:

$$(4.6) \quad \|\psi(t)\|_{H^k} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} \rightarrow \infty, \quad \text{as } t \rightarrow t^*.$$

Consider $\psi(t), n(t), n_t(t)$ posed at some time $t \in [0, t^*)$. If for some M

$$(4.7) \quad C(\|\psi(t)\|_{H^k} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}}) + C(T-t)^{\theta_\ell} M^2 \leq M$$

then $T < t^*$. Therefore, $\forall M > 0$,

$$(4.8) \quad C(\|\psi(t)\|_{H^k} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}}) + C(t^*-t)^{\theta_\ell} M^2 > M.$$

Choosing $M = 2C(\|\psi(t)\|_{H^k} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}})$ we have

$$(4.9) \quad M > C(t^*-t)^{-\theta_\ell},$$

equivalently,

$$(4.10) \quad \|\psi(t)\|_{H^{\ell+1/2}} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} > C(t^*-t)^{-\frac{1}{4}(1+2\ell)^-}.$$

which concludes the proof of Theorem 1.1.

REFERENCES

- [1] H. Added, S. Added, *Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension deux*, C.R. Acad. Sc. Paris A **299** (1984), 551–554.
- [2] I. Bejenaru, S. Herr, *Convolutions of singular measures and applications to the Zakharov system*, J. Funct. Anal., **261** (2011), 478–506.
- [3] I. Bejenaru, S. Herr, J. Holmer, D. Tataru, D., *On the 2D Zakharov system with L^2 Schrödinger data*, Nonlinearity, **22** (2009), 1063–1089.
- [4] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), 107–156.

- [5] J. Bourgain, *On the Cauchy and invariant measure problem for the periodic Zakharov system*, Duke Math. J. **76** (1994), 175–202.
- [6] J. Bourgain, J. Colliander, *On wellposedness of the Zakharov system*, Int. Math. Res. Notices **11** (1996), 515–546.
- [7] O.B. Budneva, V.E. Zakharov, V.S. Synakh, *Certain models for wave collapse*, Sov. J. Plasma Phys. **1** (1975), 335–338.
- [8] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, **10**, 2003.
- [9] T. Cazenave, F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s* , Nonlinear Anal., **14** (1990), 807–836.
- [10] J. Colliander, J. Holmer, N. Tzirakis, *Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems*, Trans. Amer. Math. Soc. **360** (2008), 4619–4638.
- [11] L. Glangetas, F. Merle, *Existence of self-similar blow-up solutions for Zakharov equation in dimension two, Part I*, Comm. Math. Phys. **160** (1994), 173–215.
- [12] J. Ginibre, Y. Tsutsumi, G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151**(1997), 384–436.
- [13] J. Holmer, *Local ill-posedness of the 1D Zakharov system*, Electron. J. Differential Equations **24** (2007), 22 pp.
- [14] M.J. Landman, G.C. Papanicolaou, C. Sulem, P.-L. Sulem, X.P. Wang, *Stability of isotropic self-similar dynamics for scalar-wave collapse*, Phys. Rev. A **46** (1992), 7869–7876.
- [15] F. Merle, *Lower bounds for the blow-up rate of solutions of the Zakharov equation in dimension two*, Comm. Pure Appl. Math. **49** (1996), 765–794.
- [16] F. Merle, *Blow-up results of virial type for Zakharov equations*, Comm. Math. Phys. **175** (1996), 433–455.
- [17] T. Ogawa, Y. Tsutsumi, *Blow-up of H^1 solution for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. **111** (1991), 487–496.
- [18] H. Pecher, *Global well-posedness below energy space for the 1-dimensional Zakharov system*, Internat. Math. Res. Notices, **19** (2001), 1027–1056.
- [19] S. Schochet, M. Weinstein, *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence*, Comm. Math. Phys., **106** (1986), 569–580.
- [20] C. Sulem, P.-L. Sulem, *Quelques resultats de régularité pour les équations de la turbulence de Langmuir*, C.R. Acad. Sc. Paris **289** (1979), A 173–176.
- [21] C. Sulem, P.-L. Sulem, *The Nonlinear Schrödinger equation: Self-focusing and wave collapse*, Series in Applied Mathematical Sciences, **139**, Springer, 1999.
- [22] N. Tzvetkov, *Low regularity solutions for a generalized Zakharov system*, Differential Integral Equations, **13** (2000), 423–440.
- [23] F.B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math., **38**(1981), 29–40.
- [24] V.E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP, **35** (1972), 908–914.
- [25] V.E. Zakharov, L.N. Shur, *Self-similar regimes of wave collapse*, Sov. Phys. JETP **54** (1981), 1064–1070.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO

DEPARTMENT OF MATHEMATICAL SCIENCES, BINGHAMTON UNIVERSITY (SUNY)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO